Lie point-symmetries for autonomous systems and resonance



1992 J. Phys. A: Math. Gen. 25 1535

(http://iopscience.iop.org/0305-4470/25/6/014)

J. Phys. A: Math. Gen. 25 (1992) 1535–1538. Printed in the UK

# Lie point-symmetries for autonomous systems and resonance

G Cicogna† and G Gaeta‡

† Dipartimento di Fisica dell'Universita', piazza Torricelli 2, 56126 Pisa, Italy
‡ IHES, 91440 Bures-sur-Yvette and CPTh, Ecole Polytéchnique, 91128 Palaiseau, France

Received 28 May 1991

**Abstract.** We give some result on the problem of finding the Lie point-symmetries of autonomous systems of differential equations. In particular, we consider here the case in which the nonlinear terms are resonant (in the sense of the Poincaré procedure for reducing the system to normal form), and we show that Lie symmetries can be characterized in a useful form.

In a previous paper [1], we pointed out the existence of a close relationship between the classical Poincaré procedure for reducing a nonlinear system of ordinary differential equations into normal form [2] and the problem of finding the Lie point-symmetries [3–5] admitted by the system. The purpose of this paper is to provide some further result related to the same argument.

With the same notation as in [1], we will consider autonomous differential systems of the following form, with $u = u(t) \in R^n$,

$$\dot{u} = f(u) = Lu + h(u) \tag{1}$$

where $f: \Omega \to R^n$ is assumed to be an analytic vector field, with $0 \in \Omega \subset R^n$, $f(0) = 0$, and $L = \partial_u f(0)$ is the linear part of $f$. We will be concerned with time-independent Lie point-symmetries admitted by (1): they are generated by operators of the form (for details see [1, 3–6])

$$\eta = \varphi_i(u)\frac{\partial}{\partial u_i} \equiv \varphi(u)\partial_u \qquad \text{(sum over } i = 1, \ldots, n) \tag{2}$$

and the determining equation for the functions $\varphi(u)$ is

$$\{\varphi, f\} = 0 \tag{3}$$

where the (Poisson) bracket is defined by

$$\{\varphi, f\}_j = \varphi_i \frac{\partial f_j}{\partial u_i} - f_i \frac{\partial \varphi_j}{\partial u_i}. \tag{3$'$}$$

In this paper we will consider the case in which all the nonlinear terms $h(u)$ in (1) are resonant [2]: this is an interesting case, because it is known that, due to the Poincaré–Dulac theorem [2], any system (1) can be converted, by a formal or converging series, into a system containing only resonant terms. The condition for the terms $h$ to be resonant with $L$ can be written [2, 7]

$$(D_{L^*}h)_j \equiv L^*_{ji} h_i - L^*_{ik} u_k \frac{\partial h_j}{\partial u_i} = 0 \tag{4}$$

or, in compact form,

$$D_{L^*}h \equiv (L^* - (L^*u)\cdot\partial)h = 0 \tag{4'}$$

where $L^*$ is the adjoint of $L$, and $D_{L^*}$ is the 'homological operator' associated with $L^*$. It has been shown [7] that if the nonlinear terms $h$ are resonant with $L$, then they admit the linear symmetry generated by $L^*$, which is

$$\eta_{L^*} = (L^*u)\partial_u.$$

Unfortunately, the linear part $Lu$ of (1) does not possess this symmetry (unless $[L, L^*]=0$, which just implies $L$ diagonalizable; see also [1, proposition 4] for the case of diagonal $L$; notice that in [7] only linear symmetries are dealt with). Using this argument, we can say:

*Proposition 1.* If all nonlinear terms $h(u)$ in (1) are resonant with $L^*$, i.e. if they satisfy

$$D_L h = 0 \tag{5}$$

(both if $L$ is diagonalizable or not), then the system (1) admits the linear symmetry $\eta_L$ generated by $L$, i.e.

$$\eta_L = (Lu)\partial_u. \tag{6}$$

If there is some non-zero constant matrix $\Phi$ (not a multiple of $L$), commuting with $L$ and such that $D_\Phi h \equiv (\Phi - (\Phi u)\cdot\partial)h = 0$, then

$$\eta_\Phi = (\Phi u)\partial_u$$

is another linear symmetry for (1).

*Proof.* It is sufficient to observe that, with $\varphi(u) = \Phi u$, the determining equations (3) become

$$[L, \Phi]u - D_\Phi h = 0. \qquad \square$$

Other (nonlinear) Lie-point symmetries for (1) in the case of resonant terms can be characterized in the form shown in proposition 2 below. In view of this, let us state the following result, which may be of some independent interest, and whose proof requires just some short calculation.

*Lemma.* Given $L$, if $h(u)$ and $g(u)$ are two vector fields resonant with $L^*$ (i.e. $D_L h = D_L g = 0$), then also

$$p(u) = \{h, g\}$$

(where the bracket $\{\cdot, \cdot\}$ is defined as in (3')), is resonant with $L^*$.

*Proposition 2.* Assume in (1) that all the nonlinear terms $h(u)$ are resonant with $L^*$ (or respectively with $L$ if $L$ is diagonalizable): then, there exist time-independent Lie point-symmetries $\eta = \varphi(u)\partial_u$ admitted by (1), such that the vector functions $\varphi(u)$ are resonant with $L^*$ (or respectively with $L$). These symmetries $\eta$ are also admitted by the system

$$\dot{u} = h(u) \tag{7}$$

obtained from (1) dropping its linear part $Lu$. Conversely, if

$$\eta' = \varphi'(u)\partial_u$$

is a symmetry for (7), and if $\varphi'$ are resonant with $L^*$ (i.e. they satisfy $D_L\varphi' = 0$), then $\eta'$ is a symmetry also for the system (1).

*Proof.* The determining equations (3) can now be written

$$D_L\varphi = \{h, \varphi\}.$$

The solution $\varphi_0(u) = Lu$ corresponds to the linear symmetry $\eta_L$ (6). In order to find other solutions, the above lemma shows that one may restrict the problem to the linear subspace of those terms $\varphi$ which are resonant with $L^*$: i.e. $D_L\varphi = 0$, then one remains with $\{h, \varphi\} = 0$. Conversely, consider now the system (7): its Lie symmetries $\eta'$ are determined by the equation

$$\{\varphi', h\} = 0. \tag{8}$$

If now one can find among the solutions of (8) some $\varphi'$ which is resonant, i.e. which satisfies also

$$D_L\varphi' = 0 \tag{9}$$

then $\eta' = \varphi'\partial_u$ is also a symmetry for the initial problem (1). □

*Remark 1.* Let us note that the above result may be useful in practice: in fact, it may be simpler to solve (8), where the linear terms are dropped, than (3), as the example below will show.

*Remark 2.* In order to satisfy (9), together with (8), it may be useful to recall that, once a solution $\varphi'$ of (8) has been found, then also $\varphi'' = k(u)\varphi'$ is a solution, where $k(u)$ is any 'constant of motion' of (7), i.e. $k(u)$ satisfies

$$h_i \frac{\partial k}{\partial u_i} = 0.$$

This implies that both $\eta' = \varphi'\partial_u$ and $\eta'' = \varphi''\partial_u$ are symmetries for (7).

*Example.* Let $u \equiv (x, y, z) \in R^3$, and consider the system, where the linear part $L$ is not diagonalizable and the nonlinear terms satisfy $D_{L^*}h = 0$ (cf [7]):

$$\dot{x} = x^2 \qquad \dot{y} = x + xy \qquad \dot{z} = y + xz \tag{10}$$

It is immediate to verify that $\eta_L = (Lu)\partial_u$ is a symmetry for (10), as expected. It is also simple to see that the system $\dot{u} = h(u)$ where the linear terms are dropped, admits, apart from the (obvious) symmetry generator describing the dynamical flow [6]

$$\eta' = \eta_h = x^2\partial_x + xy\partial_y + xz\partial_z \equiv h(u)\partial_u$$

the scaling symmetries

$$\eta_1' = y\partial_y \qquad \text{and} \qquad \eta_2' = z\partial_z. \tag{11}$$

None of the symmetries (11) is a symmetry for the initial problem (10). To obtain symmetries for (10), taking into account remark 2, one has to multiply them by a suitable function of the time-independent constants of motion $k(x, y, z)$ of the problem $\dot{u} = h(u)$, which are

$$k_1 = \frac{x}{y} \qquad k_2 = \frac{y}{z}$$

in such a way that the new symmetry $\eta' = \varphi'\partial_u$ satisfies the resonance condition $D_L\varphi' = 0$. Proceeding in this way from (11), one obtains for instance the following symmetry also admitted by (10)

$$\eta = \left(\frac{y^2}{x} - 2z\right)\partial_z.$$

*Remark 3.* An obvious solution of both (8) and (9) is $\varphi' = h$, giving $\eta' = \eta_h = h\partial_u$, which is in fact a nonlinear symmetry admitted by the initial system (1) (and by (7), of course). Notice that the combination $\eta_L + \eta_h$, where $\eta_L$ is the linear symmetry (6), gives just the symmetry $\eta_f = f(u)\partial_u$ which is the generator of the dynamical flow of (1).

*Remark 4.* Equations (3) and (8) can be solved either by the method of characteristics, or by the Ovsjannikov procedure [3, 6]. It can be noted that, as in the Example above, this procedure does not require that the vector functions $\varphi(u)$ are expressed as polynomial expansions. In the case, inspired by Poincaré method, that one assumes $\varphi(u)$ of the form

$$\varphi(u) = \Phi u + \psi(u) = \Phi u + \sum_{m>1} \psi^{(m)}(u)$$

where $\psi^{(m)}(u)$ are combinations of monomials of degree $m$, then the determining equations (3) become

$$\begin{aligned} &[\Phi, L] = 0 \\ &D_L \psi = D_\Phi h + \{h, \psi\}. \end{aligned} \tag{12}$$

Writing $h(u) = \Sigma_{(m)} h^{(m)}(u)$, the second of these can be solved step by step for each $m > 1$, i.e. [1, 6]

$$D_L \psi^{(m)} = D_\Phi h^{(m)} + \sum_{(m)} \{h^{(a)}, \psi^{(b)}\} \tag{12$'$}$$

where the sum is extended to all possible brackets giving monomials of degree $m$, thus obtaining a (formal, or possibly converging) series. In particular, if $\Phi$ commutes with $L$ and $D_\Phi h = 0$ (possibly $\Phi = 0$), and $\psi(u)$ is as in proposition 2, then $(\Phi u + \psi)\partial_u$ is a symmetry for (1).

**Acknowledgments**

Our interest in the connections between Lie point-symmetries and normal forms was stimulated by interesting conversations with Professor D Bruno during his stay at IHES. The work of GG was supported by a CNR fellowship under grant 203.01.48.

**References**

[1] Cicogna G and Gaeta G 1990 *J. Phys. A: Math. Gen.* **23** L799
[2] Arnold V I 1983 *Geometrical Methods in the Theory of Ordinary Differential Equations* (New York: Springer)
[3] Ovsjannikov L V 1962 *Group Properties of Differential Equations* (Novosibirsk, Engl transl. 1967 by G Bluman); 1982 *Group Analysis of Differential Equations* (New York: Academic)
[4] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)
[5] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (Berlin: Springer)
[6] Cicogna G and Gaeta G 1990 Lie point symmetries in bifurcation problems *Ann. Inst. H Poincaré* in press
[7] Elphick C, Tirapegui E, Brachet M E, Coullet P and Iooss G 1987 *Physica* **29D** 95